BIONETS

WP 2.2 – PARADIGM APPLICATIONS AND MAPPING

D2.2.5 Application of Game Theory and Statistical Physics Models to BIONETS Architecture, Protocols and Services

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SUMMARY

The document is concerned with paradigms in game theory, statistical physics and their potential applications to BIONETS architecture, protocols and services. It presents both our theoretical contributions to those paradigms as well as alternative contributions that mainly apply to these paradigms.

The deliverable contains 6 chapters; two on game theory and four on physics and related paradigms. The game theory part includes contributions to the fundamentals of game theory which we have developed for the context of decentralized non-cooperative decision making in large complex systems. The motivation is to adapt game theory to autonomous networks of which BIONETS is a representative. We describe in the deliverable a rich number of fundamental contributions as well as actual scenarios in wireless networks that make use of them.

In Section 2.3, the issue of cooperation of U-nodes in retrieval and dissemination of information from T-nodes is examined in two specific application models. We consider both cases where the members of a set of U-nodes have conflicting or coinciding preferences for T-node information objects. In the first case, we study incentives necessary for cooperation, which are based on bilateral object exchanges. In the second case, we propose a scheme for distribution of retrieval tasks among the U-nodes that have the same preferences. This application study is related to research conducted in WPs 1.2 and 1.3 for algorithm design and evaluation, as all strategies can be applicable in designing protocols for information retrieval and dissemination in a BIONETS environment.

Interestingly, this report is one of the first attempts in the literature to show how methods borrowed from physics (mainly random matrix theory, free probability and entropy methods) can be used for the analysis and design of wireless networks. When the dimensions of wireless networks grow, nodes can be modelled as particles which interact in a large physical system. These physical systems are known to behave according to certain deterministic laws. In the same vein, we show that BIONETS also behave according to some predictable laws for which we are able to extract the parameters of interest: density of nodes, signal to noise ratio, distribution of the nodes deployment, to name a few examples. We also introduce a whole new theory named free deconvolution (based on free probability) which enables BIONETS to extract information on the network based on a few observables quantities.

Similarly to the particle system abstraction, road traffic analysis is used to describe the behaviour of such large wireless networks. The paradigm from road traffic engineering is concerned with routing in highly dense ad-hoc networks where the density is so high that one may approximate the nodes and links by a continuous spatial density. This problem has been previously studied using tools from electrostatics and optics. Another adaption of physics is the use of potential theory which turns a problem of individual optimization by infinitely many sources into an equivalent one in which there is a single decision maker. It applies to **static** ad-hoc networks that contain a very large number of forwarding nodes.

Contents

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1 Contribution to the Foundations of Game Theory and Applications

1.1 Control of weakly coupled Markov Chains

Non-cooperative games deal with a situation of several decision makers (often called agents, users or players) where the cost of each one of the players may be a function of not only its own decision but also of decisions of other players. The choice of a decision by any player is done so as to minimize its own individual cost.

Non-cooperative games also allow one to model sequential decision making by non-cooperating players. They allow one to model situations in which the parameters defining the games vary in time. The game is then said to be a *dynamic game* and the parameters that may vary in time are the *states* of the game. At any given time (assumed to be discrete) each player takes a decision, also called an *action*, according to some predefined strategy. The vector of actions chosen by players at a given time, called a *multi-action*, may determine not only the cost for each player at that time; it can also determine the state evolution. Each player is interested in minimizing some functions of all the costs or rewards at different time instants. In particular, we shall consider here the expected time-average costs for the players.

We hereafter introduce a new class of stochastic decentralized games which we call "cost coupled constrained stochastic games" which are well adapted to decentralized scenarios typical to BIONETS. These games are characterized by the following:

1. We associate to each player a Markov chain, whose transition probabilities depend only on the action of that player,
2. We assume that at any time, each player has information only on the current and past states of his own Markov chain as well as of his previous actions. He does not know the state and actions of other players.
3. Each player has constraints on his strategies (to be defined later). We consider the general situation in which the constraints for a player depend on the strategies used by the other players.
4. Cost functions are defined (one per player) that depend on the states and actions of *all* players; each player wishes to minimize its own cost.

The players "interact" only through the last two points above.

Identifying policies that lead to equilibrium states (even in absence of constraints) is a hard problem. Unlike the situation in Markov Decision Processes (MDPs) in which stationary optimal strategies are known to exist (under suitable conditions), and unlike the situation in constrained MDPs (CMDPs) with multichain structures, in which optimal Markov policies exist, equilibrium strategies in stochastic games depend in general on the whole system past history. In other words, the amount of information required to take a decision grows in time and thus requires an unbounded amount of memory[1]

The difficulty of exhibiting strategies that lead to equilibrium states has motivated researchers to search for various possible structures of stochastic games in which saddle point policies exist among stationary

[1] An example of a simple game in which this is the case is known as the "Big Match Game", Gillette (1957).

or Markov strategies and are easier to compute. In line with this approach, we have identified conditions under which constrained equilibria exist for cost-coupled constrained stochastic games. Our theoretical contributions have appeared in [2].

We have applied these modeling techniques to wireless applications in [1] and were able to propose efficient solution methodologies.

1.2 Single controller Markov Game

A way of describing a Markov game is to consider multiple matrix games, each identified with a "state". At each time instant one of the games is played; several players choose simultaneously their actions (e.g. the row and the column of a matrix), which determine payoffs for each player. The state evolution is given by a controlled transition probability matrix. It has the following Markovian property: the next state (i.e. the identity of the next game to be played) conditioned on the present state and on the present actions of the players does not depend on the past states and actions. In that sense, Markov games originate from and extend Markov chains.

This does not mean that the state process is Markovian, since some dependence of future states on all the past history given the present state can exist through the way actions are selected. A dependence can be brought in if the decision rule for choosing an action at a given time has a dependence on the whole history. One of the central research issues in Markov games has been to identify special structures of games that guarantee the existence of Markovian or stationary equilibrium policies. A Markovian policy chooses an action a according to a probability law that is a function of only the current state and time. The choice of a is independent on any previous action or state. A stationary policy for a player is a Markovian policy in which choices of actions do not depend on time. Under any Markov policy, the state process of the Markov game is a Markov chain. Under a stationary policy, this Markov chain is, moreover, time-homogeneous.

In [13] we study a zero-sum Markov game, in which two players have opposite objectives. We restrict to the case in which only one player controls the transition probabilities. We further introduce side constraints on both players. We propose a linear program approach for deriving optimal policies.

As a potential application, consider the following jamming problem. Mobile 1 transmits streaming traffic to some destination. A second mobile transmits in the same band in order to create interference. Both mobiles apply power control and are subject to average power constraints. The channel state process of mobile i at time t is described by a Markov chain X_t^i. Mobile 1 controls also the transmission rate taking into account his buffer size (modelled in [1]). His objective is to minimize packet loss at the buffer and maximize the transmission rate of packets (i.e. the throughput). This model falls into the framework described above; indeed, we note that mobile 2 does not control his Markov chain - he only controls its power.

1.3 Evolutionary Games

Evolutionary games have been developed by J. Maynard-Smith to model the evolution of population sizes as a result of competition between them that occurs through many local pairwise interactions, i.e. interactions between randomly chosen pairs of individuals. Central in evolutionary games is the concept of Evolutionary Stable Strategy (ESS), which is a distribution of (deterministic or mixed) actions such that

if used, the population is immune against penetration of mutations. This notion is stronger than that of Nash equilibrium as ESS is robust against a deviation of a whole fraction of the population whereas the Nash equilibrium is defined with respect to possible deviations of a single player. A second foundation of evolutionary games is the replicator dynamics that describes the dynamics of the sizes of the populations as a result of the fitness they receive in interactions. Maynard-Smith formally introduced both, without the need of an explicit modeling of stochastic features. We shall call this the deterministic evolutionary game.

Randomness is implicitly hinted in the requirement of robustness against mutations, and indeed the ESS is defined through robustness against any mutation. Random aspects can be explicitly included in the modeling of evolutionary games. We first note that since deterministic evolutionary games deal with large populations, they may provide be used as a deterministic approximation of the average behaviour of smaller random games, consistently with some law of large numbers.

1.3.1 Markov Decision Evolutionary Games

In [10], we introduce a new class of stochastic evolutionary games, which we call "Markov Decision Evolutionary Games" (MDEG). There are again many local interactions among individuals belonging to large populations of players. Each individual stays permanently in the system; from time to time this individual moves among different states, and interacts with other users. The actions of the player along with those with which he interacts determine not only the immediate fitness of the player but also the transition probabilities to his next state. Each individual is thus faced with an MDP in which he tends to maximize the expected average cost criterion. Each individual knows only the state of his own MDP, and does not know the state of the other players he interacts with. The transition probabilities of a player's MDP are only controlled by himself. The local interactions between players can be viewed as a cost-coupled stochastic game [1, 2] which suggests the sufficiency of stationary strategies.

As an application of an MDEG to mobile communications, we consider in [10] the case in which individuals have finite life time and the criterion that is maximized is the total expected fitness during the individual's life time. The wireless communication scenario is as follows: mobile terminals transmit packets occasionally. Their destinations occasionally receive simultaneously a transmission from another terminal: this results into a collision. It is assumed however that even when packets collide, one of the packets can be received correctly if transmitted at a higher power. The immediate fitness rewards successful transmissions and penalizes energy consumption. Each mobile decides at each slot what his power level will be. This decision is allowed to depend on the depletion level of the battery, which serves as the "individual state". The battery is considered to be either in the state "Full" (F) in which case there are two power levels available: high and low, or "Almost Empty" (AE) in which case only the weak power level is available, or at the empty state E. Transmission at high power at state F results in a larger probability of moving to state AE. When at state E, the battery is replaced by a new one at some constant cost.

1.3.2 Non-reciprocal interactions and random number of players

There has been a lot of work on non-cooperative modeling of power control using game theory, but none using evolutionary games. There are two advantages in doing so within the framework of evolutionary games:

- it provides a stronger notion of equilibria, the ESS, which allows us to identify robustness against deviations of more than one mobile, and

- it allows us to apply the generic convergence theory of replicator dynamics, and stability results that we introduce in future sections.

Contribution of our work

The contribution of our work in [20, 21] can be summarized in three points:

- The first objective is to extend the evolutionary game framework to allow for an arbitrary (possibly random) number of players involved in a local interaction (possibly non-reciprocal interaction);

- The second objective is to apply the aforementioned extended model to access games that extend to more than two interacting nodes. In the context of Medium Access Games, we study the impact of the node distribution in the game area on the equilibrium stable strategies of the evolutionary game. The interaction between more than two individuals in a population is a new concept in evolutionary game theory and has a lot of application in multiple access games for wireless networks. Considering this kind of games, we use the notion of expected utility as this game is not symmetric. Indeed the number of players with which a given one interacts may vary from one to another; and also non-reciprocity property. We consider the following parameters in the access game: transmission cost, collision cost and regret cost. We analyze the impact of these parameters on the probability of successful transmission and give some optimization results. The notion of *correlated evolutionary stable strategy* and coordination mechanism are used to improve the performance (through collision reduction).

- The third objective is to apply evolutionary game models to study the interaction of numerous mobiles competing in a wireless environment. The power control game in wireless networks is a typical non-cooperative game where each mobile decides his transmit power in order to optimize his performance. This application of non-cooperative game theory is studied in several previous articles. The main difference in this work is the use of the evolutionary game theory which deals with population dynamics that is well adapted for studying power control games in dense wireless networks. Specifically, we focus our first study on a power control game in a dense wireless network where each user transmits using orthogonal codes, such as those used in W-CDMA. In a second scenario, we consider also uplink transmissions but inter-cell interferences like in WiMAX cells deployment. The utility function of each mobile terminal is based on carrier (signal)-to-interference ratio and pricing scheme proportional to transmitted power. We provide and predict the evolution of population between two types of behaviors : aggressive (high power) and peaceful (low power). We identify cases in which at ESS, only one population prevails (ESS in pure strategies) and others, in which an equilibrium between several population types is obtained. We also provide conditions of the uniqueness of ESS. Furthermore, we study different pricing for controlling the evolution of population.

1.4 Signalling games with application to jamming

We consider in [17] a system with two entities. The system state is a random vector of dimension n. At any given time instant the first entity (the *controller*) has complete information about the system state and must reveal a "minimum" amount of information about the system state to the second entity. It can choose the nature of the information it reveals, so as to satisfying the above constraint. The second entity (the *actor*) takes certain actions in response to the information the controller reveals; these actions are associated with certain utility functions for both the controller and the actor, which also depend on the system state. Any action and system state may lead to different utility functions for the controller and the actor, and usually when one entity receives a high reward the other receives a low reward. We devise a framework that enables the controller to decide the information it would reveal, or conceal, so as to maximize its own utility, and the actor to determine its responses to the information the controller reveals so as to again maximize its own utility.

Motivation: Jamming by concealing information

Consider a transmitter with access to n channels, the fading qualities of which constitute the state of the system. The transmitter needs to select one channel for transmission, whose quality determines the achievable rate of successful transmissions. Hence, the transmitter probes the channels in order to assess their qualities before to transmit any packet. A malicious entity, call it a jammer, seeks to reduce the rate of successful transmissions. The jammer is usually assumed to accomplish its goal by generating signals that interfere with transmitter's communication; however the jammer may be able to deteriorate the transmission rate much more by preventing the transmitter from learning the states of the channels. This may cause the transmitter to make a wrong choice, that is, select a channel with a poor transmission quality, and thereby achieve a poor data rate for a while. Note that the jammer can prevent the transmitter from learning the channel states, possibly by generating signals that interfere the probe packets or responses to these probes, and generating such signals may consume less energy as compared to those that jam the actual transmission since the probe packets are transmitted over shorter durations. We therefore consider the case where the jammer blocks the probe packets and not the actual transmission. Furthermore, we assume that the jammer knows the quality of the channels and can block the probes in at most k channels since the blocking process consumes energy. Hence, the states of at most k channels can be concealed from the transmitter. The transmitter selects the channel after learning the states of the channels the jammer does not conceal. Note that the transmitter may either select a channel whose state has been revealed or one whose state has been concealed; the latter may happen since the fact that the jammer has concealed the state of a channel may indicate that the transmission quality of the corresponding channel is good. The rate of successful transmission attained by the transmitter determines the utility of the transmitter and the jammer. The information concealing problem we described will enable the jammer (controller) to optimally determine which channels it would conceal, and the transmitter (actor) to select the seemingly best channel.

Contribution and Challenges

Our first contribution is to provide a framework for investigating information concealing problems. We formulate this problem as a stochastic leader-follower game. We demonstrate that the well-known Nash equilibrium solution concept cannot be effectively used in this game since the utilities of the players turn out to be functions rather than scalars. Subsequently, we develop suitable solution concepts, that of point-wise Nash equilibria, that capture the subtleties of this game. For example, the actor can learn about the system not only from the information the controller reveals, but also from the choices of the controller regarding which sources of information it conceals, since the fact that an information has been concealed may provide important insight about its nature. Thus, the actor must determine its optimal action so as to exploit the information contained in both of the above, and the controller must determine what it should conceal considering that the actor will learn from both of the above. In cognitive radio networks, a naive policy for the jammer would be to conceal the states of the channels that have the k best transmission qualities. But, if the transmitter knows the jammer's policy, then it knows that the transmission quality of any channel whose state has been concealed is at least as good as that of a channel whose state has been revealed, and thus, its best action is to select a channel whose state has been concealed. But, if the jammer reveals the states of some channels whose transmission qualities are better than those whose states it conceals, the transmitter will be confused regarding the choice of the channel even when it knows the jammer's policy, and is therefore more likely to make a poor selection. Our framework formally establishes that the naive policy described above is suboptimal for a controller.

We next prove that there is a one-to-one correspondence between the set of point-wise Nash equilibria in the above game and the set of saddle points in a two-person zero-sum game turns out to be very useful as it implies that a point-wise Nash equilibrium exists for the original game and can be computed using a linear program. The equivalence is however somewhat surprising as the controller and actor have different amount of information about the system, that is, the controller has complete information whereas the actor only has partial information about the system state. Since the policies and the utilities of each player depends on the information they have about the system, the utilities of the two players turn out to be functions with different domains, and hence their sum cannot be defined, whereas the sum of the utilities in a two-person zero-sum game is always zero. Furthermore, both players act simultaneously in two-person zero-sum games, whereas in the information concealing problem, the players act sequentially: the controller first needs to reveal information about the system state, and then the actor can determine its actions.

We next investigate the computational aspects of the information concealing games. Our results in this area constitute our second contribution since general results that can address the computational aspects in this case are not known in the game theory or approximation algorithm literature. We first observe that the number of variables and constraints in the standard linear program formulations for computing the saddle points of the equivalent games are super-exponential in n, where n is the dimension of the state-space of the system. Thus, the linear program becomes computationally intractable even for moderate values of n. Exploiting specific characteristics of the game under consideration, we next obtain linear programs which compute the saddle points of the equivalent game and the optimal policies for both players while using an exponential number of variables and constraints. This significant reduction in computation time enables the

computation of the optimal policies for moderate n. We next obtain linear time ($O(n)$) computable policies with performance guarantees for both players. Specifically, these policies attain utilities that differ from the utilities of the saddle points by (a) constant factors in several important special cases, and (b) by factors that depend only on the amount of information that the controller reveals to the actor, and do not depend on n in the most general case.

2 Applications of Game Theory

2.1 Applications of Evolutionary Games

2.1.1 Applications to protocol design

The possibilities to freely deploy new versions of protocols creates a competition environment between protocols. Much work has been devoted to analyze such competition and to predict its consequences. The two main approaches for predicting whether one version of a protocol would dominate another are

- **Inter Population Competition (IRPC):** One examines local interactions between connections of different protocols that interact with each other (by sharing some common bottleneck link). If a connection that corresponds to a given protocol performs better in such an interaction, then the DC approach predicts that it would dominate and that the other protocol would vanish.
- **Intra Population Competition (IAPC):** In this approach one studies the performance of a version of a protocol assuming a world where all connections use that version. This is repeated with the other version. One then predicts that the version that gives a better world would dominate.

We address the dominance question with the evolutionary game paradigm and provide a detailed analysis of this competition scenario. Our approach predicts whether one can expect one protocol to dominate the other or whether both protocols can be expected to coexist. We also compute the share of the population that is expected to use each protocol version in the scenario when both versions coexist. Finally, it provides a description of the dynamics of the competition, which may result in a stable behavior that consists of a convergence to some equilibrium, or it may display instabilities and oscillations. By identifying the conditions for a stable behavior, one can provide guidelines for upgrading protocols so as to avoid undesirable oscillating behavior.

In [11], our first objective is to provide a framework to describe and predict the evolution of protocols in a context of competition between two types of behaviors: aggressive and peaceful.

We compute the Evolutionary Stable Strategies (ESS) for congestion protocols of different degree of aggressiveness. We identify cases in which at ESS, only one population prevails (ESS in pure strategies) and others, in which an equilibrium between several population types is obtained. To this end, we map the problems, whenever possible, onto the Hawk and Dove Game. We then study the convergence of the replicator dynamics to it.

The second objective of the paper is to provide a framework for controlling evolutionary dynamics (changing or upgrading protocols) through the choice of a gain parameter governing the replicator dynamics. We address the following two design issues concerning this choice:

(i) the trade-off between fast convergence and stability. We identify a simple threshold on the gain parameter in the replicator dynamics such that the stability is only determined by whether we exceed or not the threshold.

(ii) the stability as a function of delays. We derive new stability conditions for the replicator dynamics in the Hawk and Dove game with non-symmetric delays and apply it to the evolution of the MAC and transport layer protocols.

2.1.2 Association games

In cognitive terminal-centered radios, a mobile terminal has to decide which access points (APs) to connect to. Users scan the wireless channels in order to find the AP with the highest signal strength and associate to it. They then transmit at different rates (called the PHY rate) based on the advertised signal strength. It has been observed when using IEEE 802.11 WLANs, that all the connections in a single cell receive the same throughput, leading to inefficient channel use. The question arises as to whether it might be better for a user to split traffic among visible APs, which would provide diversity from the fact that signal strength in different cells might be different and the cells might be loaded differently.

Suppose we have a geographical region divided into cells. Each cell has an AP, each using a separate channel. This is possible in 802.11 b and g as they have three non-overlapping channels. Users may observe significant signal strengths from different APs. It is possible for users with just a single wireless network interface card to associate to all the APs available to them. Users could then probabilistically divide their traffic among the different APs. The APs could charge a price for sending packets through them. The payoff that the users obtain is the difference of the throughput minus the charged price. Traffic splitting in the Internet among different Internet Service Providers (ISPs) is called multihoming and we follow the same terminology for the WLAN case.

Our objective in [18] is to design a decentralized pricing mechanism which has the property that even with selfish users trying to maximize their own payoffs, the system throughput is actually maximized.

Main Results. We consider in [18] the existing system of UDP or TCP running over 802.11, with the added provision of multihoming. The multihoming protocol would decide the time spent associated with each of the APs, while the transport functionality will be provided through (multiple) UDP or TCP sessions. We do not insist on any particular multihoming protocol, but only make the assumption that it would be rational in the sense that it would allocate more time to APs that yield higher payoffs, and less time to those that do not. We assume that the time-scale at which users switch between the APs (a few times per second) is much greater than the time-scale at which the throughput of TCP or UDP over 802.11 converge to their average values (of the order of milliseconds). The multihoming protocol chooses the APs probabilistically. In the fluid model approximating the many users regime, the ratio of fluid split among APs gives the probabilities of associating with them.

In a WLAN, since different users send at different PHY rates, they occupy the channel for different amounts of time. We propose a pricing mechanism in which users are charged based on their channel occupancy. We call this “cost price charging”. We show that under this mechanism, when all users use either TCP or UDP along with rational multihoming protocols, the system is asymptotically stable. We also show that multihoming under the cost-price mechanism is efficient, i.e., the system throughput is

maximized.

Finally, we show that when an ISP charges differentiated prices above the cost price charge in the different APs, multihoming achieves at least the same profit as unihoming. So the ISP suffers no loss by allowing its customers to multihome. We further show that even in the case of differentiated pricing, the throughput of the system as a whole is at least that of unihoming.

The methodology we use is of population games which is an extension of evolutionary games. It differs from evolutionary games in the fact that interactions can occur with a large number of users whereas in evolutionary games, local interactions are assumed to involve a pair of randomly selected users. Both methodologies have in common the replicator dynamics and the concept of Evolutionary Stable Strategies.

2.2 Routing Games

Non-cooperative routing games have long been studied in the context of road traffic in the framework of infinite number of players (drivers) where the solution concept is that of Wardrop equilibrium. In that context they can be modeled as potential games which allow one to obtain a unique equilibrium (in terms of link flows) as a solution of an equivalent (single player) optimization problem.

In the context of computer networks, non-cooperative routing has been introduced and studied with finitely many users, each of which having to decide how to split flows between various source-to-destination links. A user may correspond to a service provider that controls the routes taken by the subscribers. This type of formulation, already studied in the context of road traffic, turns out to be much more difficult and does not enjoy in general the structure of a potential game. In particular, counter examples are known for the non-uniqueness of the equilibrium. It is therefore of interest to identify conditions on the cost structure that allow to obtain a potential game in the setting of finitely many players.

In our work [12], we show that the case of linear link costs provides such conditions. We then exploit the potential game structure to obtain convergence to equilibrium of schemes based on best responses. We further exploit the potential game structure to establish the convergence of the replicator dynamics which has an evolutionary interpretation.

2.3 Cooperative Information Retrieval

In a BIONETS system, the retrieval and dissemination of static sensor (T-node) information by user mobile devices (U-nodes) can incur a significant cost. Cooperation is required in which all U-nodes retrieve and disseminate content, so that a lower cost is incurred. In a set of U-nodes, some of the nodes may have conflicting, and some coinciding interests for T-node objects. In the case of conflicting interests, incentives are needed in order for a U-node to copy unwanted T-node information. In the case where interests coincide, cooperation can be used so that each U-node copies a different subset of objects in its limited storage, and retrieves remaining objects from other U-nodes.

In [22], we have studied the case where U-nodes have conflicting interests for T-node objects. A specific modeling scenario is considered, where the network has the form of a graph. T-nodes are located on the vertices of the graph and U-nodes move along the edges according to a Markovian waypoint model (i.e., transitions from one waypoint to the next are governed by a Markov chain; a U-node moves from waypoint T_i to waypoint T_j with a probability $p(T_i, T_j)$). We decompose the problem and study possibilities

of cooperation on each leg of the graph. We say that two U-nodes coming from opposite directions and meeting somewhere on a leg, are cooperative on this leg, if and only if each one copies the content of its origin T-node, even if they are not interested in it. Satisfying equilibrium conditions for each leg and between each pair of U-nodes means that a *cooperative equilibrium* (i.e., an equilibrium in which all U-nodes are cooperative) is achieved for the whole network.

Generally, a U-node action can be of two types: a) Cooperative (C), in which case the U-node retrieves unwanted T-node content and transmits it to interested U-nodes, and b) Selfish (S), in which case the U-node does not retrieve unwanted T-node content, or does not transmit acquired T-node content. We find conditions under which the following strategy of each U-node results in an equilibrium. This strategy is made up of two actions: *Initially, a U-node is cooperative and copies unwanted content. However, if it meets a selfish U-node somewhere on a leg, it will only transmit its acquired content with a certain probability.* The latter is called "cooperation probability".

The following assumptions are made regarding the knowledge each U-node has. Each U-node knows the topology of the network, the number of other U-nodes and the distances between each pair of T-nodes. They are also aware of the information content at each T-node. Furthermore, U-nodes know that other U-nodes move also according to the Markovian waypoint model on the graph (e.g., in absence of other information, they may all assume a random walk model for each other). On the contrary, U-nodes do not know the interests of other U-nodes for information objects and have no memory of previous encounters with them.

We calculate the cost of a U-node as the delay to retrieve information content on the specific leg. This cost depends on whether U-nodes have a cooperative or selfish behavior. If two U-nodes meet somewhere on a leg and both are selfish, the cost they incur is worse than if they were cooperative. Let $\mathcal{C}_i^{(T_i,T_j)}(C|k)$ denote the expected cost of node U_i to be cooperative when k other U-nodes are cooperative. We find the minimum number k of other cooperative U-nodes such that a U-node does not lose by cooperation, i.e. such that

$$\mathcal{C}_i^{(T_i,T_j)}(C|k) < \mathcal{C}_i^{(T_i,T_j)}(S|0).$$

The analysis shows that cooperation is more beneficial than the case where all U-nodes are selfish when there are long-distance routes (legs) or when the destination T-node is visited often by U-nodes. On the other hand, a higher number of cooperative U-nodes is required to yield a benefit in the case where U-nodes are moving at a high speed, or if the cost to receive content from the origin T-node is higher.

Then we move on to find conditions under which a cooperative equilibrium exists. The equilibrium conditions depend on the probability of meeting other U-nodes on the leg, the cost for retrieving content from the origin T-node and the cooperation probability: as the cooperation probability decreases, a smaller number of nodes in the network is required for an equilibrium to be achieved. We can expect this result, since a cooperative U-node punishes more a selfish U-node by giving its acquired content with a smaller probability. As also anticipated, an equilibrium is more easily achieved for smaller information retrieval costs. A final remark is that the interaction between the meeting and cooperation probability is less intuitive: when the cooperation probability of other U-nodes is small, a higher meeting probability leads to a U-node actually having less incentive to cooperate and appearing more selfish, whereas if the cooperation probability of other U-nodes is high, it leads a U-node to actually being cooperative.

For the case where U-nodes have common interests for T-node objects, an algorithm for cooperative T-node object retrieval has been suggested, based on the analysis in [23]. We assume that each U-node has certain request rates for a number of T-node objects, and the space required by these objects exceeds its memory capacity. If a U-node stores an object in its own local memory, the cost to retrieve it has a relatively small value t_ℓ. A U-node must retrieve an object not contained in its local memory either from the origin T-node or from a U-node that has the object. The mean cost for retrieving an object from a T-node, denoted by t_s, can be taken as the mean hitting time to a T-node from a random initial position in the network. If, on the other hand, a number n of other U-nodes also have the same object, the cost for retrieving the object from a U or T-node, denoted by $t_r(n)$, can be taken as the minimum of the average intermeeting time to any of the U-nodes, or the origin T-node. Without considering a detailed underlying model, we assume in the analysis that $t_r(n)$ is a decreasing function of n, and that $t_\ell < t_r(n) < t_r(n-1) < \cdots < t_r(1) < t_s$ holds.

We consider that U-nodes with common interests for T-node objects cooperate with each other to retrieve objects by forming a *replication group*. Each U-node knows the total number of other U-nodes in the group, as well as their preferences for objects. Based on this knowledge it implements a *placement strategy*, which consists of choosing which objects to replicate locally in its limited storage space. By appropriately placing objects, an "enhanced local storage space" can be created so as to decrease the total cost per unit time of each U-node for retrieving objects.

We model this as a sequential decision game, where U-nodes play sequentially; each move consists of deciding which objects to place in their memory, based on the placements of U-nodes that played before them, and the preferences of the remaining U-nodes. We examine properties of the following *greedy* strategy: at its turn, each U-node makes a number of placements that minimize its access cost. In order to do this, a U-node j calculates the average *eviction cost* $L_{e,j}$ for each object e in its memory. This is the average increase in retrieval cost that would be incurred to the U-node if it evicted the object. Accordingly, for any of the objects i it does not store locally, it calculates the average *insertion gain*, $G_{i,j}$ which is the average decrease in retrieval cost that would occur by inserting the object. For a local capacity of C_j objects, a U-node j indexes the locally stored objects $\{e_1,\ldots,e_{C_j}\}$ in increasing eviction cost, i.e. such that $L_{e_1,j} \le L_{e_2,j} \le \cdots \le L_{e_{C_j},j}$; it also indexes objects $i_1,\ldots,i_{|I_j|}$ candidate for insertion in decreasing insertion gain, i.e. as $G_{i_1,j} \ge G_{i_2,j} \ge \cdots \ge G_{i_{|I_j|},j}$ (assuming the number of these objects is $|I_j|$.) Then the algorithm consists of making a number of changes k, replacing object e_1 by $i_1,\ldots,e_k$ by i_k, as long as $L_{e_k,j} < C_{i_k,j}$. It is intuitive that a U-node may evict an object that also exists in many other U-nodes, to insert an object that is not stored or is only stored at a few of the U-nodes in the group.

We examine game-theoretic properties of this algorithm in [23][2]. It is shown that although it may not arrive at an equilibrium, it possesses some good properties: i) the algorithm arrives at a stable state in a finite number of rounds, after which no player has a benefit by changing its placement, ii) in the homogeneous case where all U-nodes have the same object preferences and the same mobility pattern, the strategy is close to being individually rational for each player, in the sense that the mean total cost of a U-node is smaller than its cost by acting in isolation and retrieving all objects not stored locally from the respective T-nodes. Extensive numerical results on the performance of the algorithm are reported on BIONETS deliverable D1.2.3 [24], for several values of t_ℓ, t_s, and convex/concave decreasing functions

[2] In [23], we have considered a different setting where the cost t_r is fixed and the availability of nodes in the replication group is a random variable. However, the main results of the analysis in Section 3 of the paper still apply in our case.

$t_r(n)$. These show significant cost reductions for all nodes, as well as increases in the number of different objects locally stored in the nodes of the replication group.

Knowledge of the number of U-nodes in the replication group and their preferences for objects is necessary to apply the algorithm. Furthermore, each time a U-node changes its placement, all other U-nodes have to be informed of the change. Such knowledge can be acquired through distributed mechanisms, such as *gossiping* or *population protocols*. Some further issues that then arise are delays in updating placement information and estimation errors of the number of other U-nodes in the group or their placements. It is worth noting that research along these lines is also conducted within WP1.2 of the BIONETS project.

2.4 Other

We have had some other work on game theory applied to wireless networks in a cellular context. Most paper consider power control and related issues in non-cooperative settings. [3, 4, 19, 14, 15] We do not describe these work in details as they concern cellular network architectures that do not seem to be relevant for BIONETS.

Finally, we mention [16] that provides theoretical insight on non-efficiency of equilibria. This too did not find direct applications to the BIONETS context.

3 Paradigms from Road Traffic

3.1 Continuum Equilibrium and Routing in Wireless Ad Hoc Networks

An important approach to routing in wireless ad hoc network has been to design traffic dependent protocols that send packets along paths that have smallest delays. This metrics goes back to an early paper by Gupta and Kumar who show that by doing so, resequensing delays (that are undesirable in real time traffic and very harmful in data transfers using the TCP protocol) are minimized. A recent line of research has been to study such protocols in massively dense ad hoc networks that are characterized by the property that each node has many other nodes in its transmission range. We are interested here in the recent fluid limit approach in which the nodes are modeled as a continuum, and where the discrete graph describing the links and their costs is replaced by a cost density (which depends on the traffic intensity) over the plane. The reason of using such fluid limit approximations is that whereas the complexity of finding optimal routes grows with the number n of nodes, the fluid limit does not depend on n and hence the complexity of finding optimal routes in the fluid approximation does not grow with the number of nodes on the original system. Various approaches inspired by physics have been proposed starting with the pioneering work of Jacquet who used ideas from geometrical optics, and Kalantari and Shayman whose approach is based on electrostatics. Approaches based on electrostatics have also been designed in a series of papers by L. Tassiulas and S. Toumpis. The development of the original theory of routing in massively dense ad hoc networks has emerged in a complete independent way of the existing theory of routing in dense networks which have been developed within the community of road traffic engineers, introduced in 1952 by Wardrop and by Beckmann. In the literature of the road traffic community the modeling of traffic equilibrium problems for a transportation system is classified into the discrete modeling approach and the continuum modeling approach. In the discrete modeling approach, each road link within the network is modeled separately and

the demand is assumed to be concentrated at hypothetical zone centroids. This modeling approach is commonly adopted for the detailed planning and analysis of a transportation system. The continuum modeling approach is used for the initial phase of planning and modeling in broad-scale regional studies, in which the focus is on the general trend and pattern of the distribution, the travel choice of users, and on changes in these two factors in response to policy changes in the transportation system at the macroscopic level, rather than the detailed level. The fundamental assumption is that the difference in modeling characteristics, such as the travel cost and the demand pattern between adjacent areas within a network, is relatively small compared with the variation over the entire network. Hence, the characteristics in a network, such as the flow intensity, demand, and travel cost, can be represented by smooth mathematical functions.

This community further developed numerical approaches to solve the continuous approximation model through discretization. Although it may seem that one is back to the starting point with yet another discrete problem to solve, the new discrete problem is simpler, each node in it has only a small number of neighbors, and the number of nodes in the new discrete model is independent of the number of nodes in the original system. Inspired by this approach, we have studied in [5, 6, 7] routing in static ad hoc networks (e.g. sensor networks). In the first reference, we consider limitation to two directions which models the use of directional antennas. In the second work, we study the case where any general direction can be chosen at any point. The third work is numerical analysis of the omnidirectional problem.

Two types of objectives are sought in the research on routing in the road traffic context. The first is to maximize the global utility for the whole society, and the second is to find a routing configuration (called "traffic assignment") such that each transmission uses only paths with minimum costs. Configurations satisfying this property are known as "Wardrop Equilibrium", and they coincide with the solution concept used by Gupta and Kumar. We study the two types of objectives in the context of massively dense ad hoc networks. For the first objective (which corresponds to a cooperation between nodes) we use and strengthen results of Beckmann by using tools from optimization and control theory that have not been available at the middle of the last century. We further study the Wardrop equilibrium and establish conditions under which it coincides with the global optimization.

We first consider the one dimensional case in order to explain the main concepts involved in our model and how these concepts can be extended to the two dimensional case in order to obtain the optimal deployment of the relay nodes in a massively dense wireless ad hoc network. As a first approach we consider the line segment $[0,L]$, which will be the geographical reference of a network. We consider the continuous *node density function* $\eta(x)$, measured in nodes/m, such that the total number of nodes on a segment $[\ell_0,\ell_1]$, denoted by $N(\ell_0,\ell_1)$, is $N(\ell_0,\ell_1)=\int_{\ell_0}^{\ell_1}\eta(x)\,dx$. We consider as well the continuous *information density function* $\rho(x)$, measured in bps/m, generated by the sensor nodes such that i) at location x where $\rho(x)>0$ there is a fraction of data created by the sensor sources, such that the rate with which information is created in an infinitesimal area of size $d\varepsilon$, centered at position x, is equal to $\rho(x)\,d\varepsilon$, and ii) at location x where $\rho(x)<0$ there is a fraction of data received at the sensor destinations such that the rate with which information is received by an infinitesimal area of size $d\varepsilon$, centered at position x, is equal to $-\rho(x)\,d\varepsilon$. We assume that the total rate at which sensor destinations have to receive data is the same as the total rate which the data is

created at the sensor sources, *i.e.*, in the line segment $[0,L]$,

$$\int_0^L \rho(x)\,dx = 0. \tag{1}$$

Consider the continuously differentiable *traffic flow function* $T(x)$, measured in bps/m, such that its direction (positive or negative) coincides with the direction of the flow of information at point x and $|T(x)|$ is the rate at which information propagates at position x, *i.e.*, $|T(x)|$ gives the total amount of traffic that is passing through the position x. In order to conserve the information transmitted over a line segment $[\ell_0,\ell_1]$, it is necessary that the rate with which information is created over the segment is equal to the rate with which information is leaving the segment, *i.e.*, $T(\ell_1)-T(\ell_0)=\int_{\ell_0}^{\ell_1}\rho(x)\,dx$. The integral on the right hand side is equal to the quantity of information generated (if it's positive) or demanded (if it's negative) by the fraction of nodes over the line segment $[\ell_0,\ell_1]$. The expression $T(\ell_1)-T(\ell_0)$, measured in bps/m, is equal at the rate with which information is leaving (if it's positive) or entering (if it's negative) the segment $[\ell_0,\ell_1]$. This holding for any line segment, it follows that necessarily,

$$\frac{dT(x)}{dx} = \rho(x) \quad \text{for all } x \in (0,L). \tag{2}$$

The problem considered is to find the number of nodes $N(0,L)$ in the line segment $[0,L]$, needed to support the information created by the sources and received at the destinations subject to the flow conservation condition given by equation (2) and imposing that there is no flow of information leaving the network, *i.e.*, $T(0)=0$ and $T(L)=0$. Notice that in the one-dimensional case, there is no minimization problem since by using the constraints we obtain just one solution. As we will see, this will not be the case for the two-dimensional case. Within the one-dimensional case context, we further assume that the proportion of sensor nodes $\eta(x)$ in a line segment of infinitesimal size $d\varepsilon$, centered at location x, needed as relay nodes, is proportional to the traffic flow of information that is passing through that region, *i.e.*, $\eta(x)\,d\varepsilon = |T(x)|^{\alpha}\,d\varepsilon$ where $\alpha>0$ is a fixed number called *the relay-traffic constant.* Then the optimal placement of the relay nodes in the network will be given by $\eta^*(x)=|T^*(x)|^{\alpha}$, where the traffic flow function $T^*(x)$ is the optimal traffic flow function, given by the solution of the previous system of equations. Furthermore, the optimal total number of relay nodes $N^*(0,L)$ needed to support the optimal traffic flow function $T^*(x)$ in the network will be

$$N^*(\ell_0,\ell_1)=\int_{\ell_0}^{\ell_1}\eta(x)\,dx=\int_{\ell_0}^{\ell_1}|T(x)|^{\alpha}\,dx.$$

For the one-dimensional case the problem doesn't require any minimization. This will not be the case for the two-dimensional case.

Consider a grid area network D in the two dimensional plane[3] $X_1\times X_2$. Similar to the one-dimensional case, consider the continuous *information density function* $\rho(\mathbf{x})$, measured in bps/m^2, and the continuous *node density function* $\eta(\mathbf{x})$, measured in nodes/m^2. The total rate at which sensor destinations must process data is the same as the total rate which the data is created at the sensor sources, *i.e.*,

$$\int_{X\times Y}\rho(\mathbf{x})\,d\mathbf{x}=0.$$

[3]We will denote with bold fonts the vectors and $\mathbf{x}=(x_1,x_2)$ will denote a location in the two dimensional plane $X_1\times X_2$.

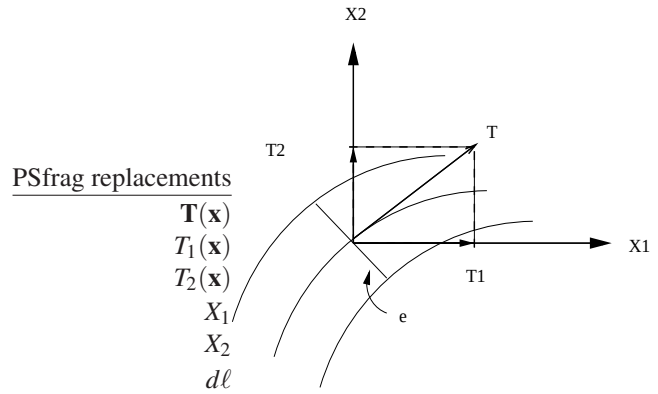

Figure 1: Flow of information $\mathbf{T}(\mathbf{x})$ through incremental line segment $d\ell$, decomposed in its horizontal component $T_1(\mathbf{x})$ (in the direction X_1) and its vertical component $T_2(\mathbf{x})$ (in the direction X_2).

The continuous *traffic flow function* $\mathbf{T}(\mathbf{x})$, measured in bps/m, such that its direction coincides with the direction of the flow of information at point $\mathbf{x}$, and[4] $\|\mathbf{T}(\mathbf{x})\|$ is the rate with which information rate crosses a linear segment perpendicular to $\mathbf{T}(\mathbf{x})$ centered on $\mathbf{x}$, *i.e.*, $\|\mathbf{T}(\mathbf{x})\|\,\varepsilon$ gives the total amount of traffic crossing a linear segment of infinitesimal length ε, centered at location $\mathbf{x}$, and placed vertically to $\mathbf{T}(\mathbf{x})$.

Next we present the flow conservation condition (see *e.g.* [8], for more details about this type of condition). For information to be conserved over a domain $\mathcal{D}$ of arbitrary shape on the $X \times Y$ plane, with smooth boundary S, it is necessary that the rate with which information is created in the area is equal to the rate with which information is leaving the area, *i.e.*,

$$\int_D \rho(\mathbf{x})\, dD = \oint_S [\mathbf{T}\cdot\mathbf{n}(\mathbf{x})]\, d\ell \tag{3}$$

The integral on the left-hand side is the surface integral of $\rho(\mathbf{x})$ over the domain $\mathcal{D}$. The integral on the right-hand side is the path integral of the inner product $\mathbf{T}\cdot\mathbf{n}$ over the boundary S. The vector $\mathbf{n}(\mathbf{x})$ is the unit normal vector to S at the boundary point $\mathbf{x} \in S$ and pointing outwards. Then the function $\mathbf{T}\cdot\mathbf{n}(\mathbf{x})$, measured in bps/m^2, is equal at the rate with which information is leaving the domain $\mathcal{D}$ at the boundary point $\mathbf{x}$. This holding for any (smooth) domain $\mathcal{D}$, it follows that necessarily

$$\nabla\cdot\mathbf{T}(\mathbf{x}) := \frac{\partial T_1(\mathbf{x})}{\partial x_1} + \frac{\partial T_2(\mathbf{x})}{\partial x_2} = \rho(\mathbf{x}), \tag{4}$$

where "$\nabla\cdot$" is the divergence operator. Notice that equations (3) and (4) are the integral and differential versions of Gauss's law, respectively.

Thus the problem considered is to minimize the quantity of nodes $N(\mathcal{D})$ in the grid area network $\mathcal{D}$ needed to support the information created by the distribution of sources subject to the flow conservation condition, *i.e.*, our problem is given by the system of equations:

$$\text{Min} \quad N(D) \tag{5}$$

$$\text{subject to} \quad \nabla\cdot\mathbf{T} = \rho(\mathbf{x}). \tag{6}$$

[4]The norm $\|\cdot\|$ is the Euclidean norm, *i.e.*, for a vector $\mathbf{x} = (x_1, x_2)$, its norm will be $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$.

Tassiulas and Toumpis prove in [9] that among all traffic flow functions that satisfy $\nabla \cdot \mathbf{T} = \rho$, the one that minimizes the number of nodes needed to support the network under Assumption 1, must be irrotational, *i.e.*,

$$\nabla \times \mathbf{T} = 0. \tag{7}$$

where "$\nabla\times$" is the curl operator.

3.2 Directional Antennas

For energy efficiency reasons, we assume that each node is equipped with one or two directional antennas, allowing transmissions at each hop to be directed either from North-to-South or from West-to-East following the notation convention used in the work of Dafermos [8]. We have the following result for the system-optimization problem (for more details see [6]):

Theorem 3.1 *Define the Lagrange function as*

$$L^{\zeta}(\mathbf{T}) := \int_{\mathcal{D}} \ell^{\zeta}(\mathbf{x},\mathbf{T})\, d\mathbf{x} \quad \textit{with} \quad \ell^{\zeta}(\mathbf{x},\mathbf{T}) := g(\mathbf{x},\mathbf{T}) - \sum_{j=1}^{m} \zeta^j(\mathbf{x}) \left[\nabla\cdot\mathbf{T}^j(\mathbf{x}) - \rho^j(\mathbf{x})\right]$$

where $\zeta^j(\mathbf{x}) \in L^2(\mathcal{D})$ are called Lagrange multipliers. For a vector field $\mathbf{T}(\cdot)$ with positive components satisfying (4), a necessary and sufficient condition for minimizing the sytem-cost is that the Lagrangian be minimized over all vector fields with positive components, or equivalently, that equations

$$\frac{\partial g(\mathbf{x},\mathbf{T})}{\partial T_i^j} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} = 0 \quad \textit{if} \quad T_i^j(\mathbf{x}) > 0, \tag{8a}$$

$$\frac{\partial g(\mathbf{x},\mathbf{T})}{\partial T_i^j} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} \geq 0 \quad \textit{if} \quad T_i^j(\mathbf{x}) = 0. \tag{8b}$$

be satisfied.

We assume that the local transmission cost depends on the direction of the flow but not on its size, *i.e.*, it is a congestion independent cost. The cost is $c_1(\mathbf{x})$ for a flow that is locally horizontal and is $c_2(\mathbf{x})$ for a flow that is locally vertical. As previously stated, we assume in this section that c_1 and c_2 do not depend on $\mathbf{T}$. The cost incurred by a packet transmitted along a path p is given by the line integral

$$\mathbf{c}_p = \int_p \mathbf{c}\cdot d\mathbf{x}. \tag{9}$$

Let $V^j(\mathbf{x})$ be the minimum cost to go from a point $\mathbf{x}$ to a set B^j, $j = 1,...,m$. Then

$$V^j(\mathbf{x}) = \min\left(c_1(\mathbf{x})\, dx_1 + V^j(x_1+dx_1,x_2), c_2(\mathbf{x})\, dx_2 + V^j(x_1,x_2+dx_2)\right). \tag{10}$$

This can be written as the Hamilton Jacobi Bellman (HJB) equation:

$$0 = \min\left(c_1(\mathbf{x}) + \frac{\partial V^j(\mathbf{x})}{\partial x_1}, c_2(\mathbf{x}) + \frac{\partial V^j(\mathbf{x})}{\partial x_2}\right), \qquad \forall \mathbf{x} \in B^j,\, V^j(\mathbf{x}) = 0. \tag{11}$$

If V^j is differentiable then, under suitable conditions, it is the unique solution of (11). In the case that V^j is not everywhere differentiable then, under suitable conditions, it is the unique viscosity solution of (11).

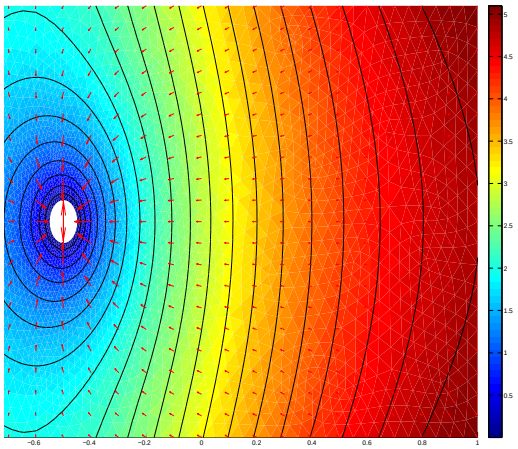

Figure 2: Solution for a wireless sensor network.

There are many numerical approaches for solving the Hamilton-Jacobi-Bellman (HJB) equation. One can discretize the HJB equation and obtain a discrete dynamic programming for which efficient solution methods exist. If one repeats this for various discretization steps, then we know that the solution of the discrete problem converges to the viscosity solution of the original problem (under suitable conditions) as the step size converges to zero.

As an example consider that the region under study can be modelled by the rectangle $[-1,1]\times[-0.5,0.5]$ and the centre of analysis of information is located at the point $(-0.5,0)$. If we suppose that we distribute uniform over the whole network, then is reasonable to suppose if the land doesn't have other environmental problems the cost of the network will be uniform and for simplicity we consider $c=1$. After using the modelling explained in the previous section the direction of the flow of information can exactly be described by the red lines in Figure 2.

3.3 Conclusions

Routing in ad hoc networks has received much attention in the massively dense limit. The main tools to describe the limits had been Electrostatics and geometric optics. We exploited another approach for the problem that has its roots in road traffic theory, and presented both quantitative as well as qualitative results for various optimization frameworks. The links to road traffic theory allow us to benefit of the results of more than fifty years of research in that area that not only provide mature theoretical tools but have also advanced in numerical solution methods.

4 Paradigms from Physics

4.1 Statistical Physics foundations

4.1.1 Historical Perspective

In the 1950's, Eugene P. Wigner, who was interested in deriving the energy levels of some heavy nuclei, made the assumption that these levels would behave as the eigenvalues of some matrices with random entries. It turns out that the energy levels are linked to the Hamiltonian operator through the following Schroedinger equation:

$$\mathbf{H}\varphi_i = \mathbf{E_i}\varphi_i$$

where
φ_i is the wave function
$\mathbf{E_i}$ is the energy level
$\mathbf{H}$ is the Hamiltonian

Hence, the energy levels of the operator $\mathbf{H}$ are nothing else than the eigenvalues of the matrix representation of that operator. For a specific nucleus, finding the exact energy level is a very intricate problem as the number of interacting particles increase. The idea of Wigner was to replace the exact matrix by a large dimensional random matrix having the same properties. Hence, in some cases, the matrix can be replaced by the following Hermitian random matrix where the upper diagonal elements are i.i.d., generated with a binomial distribution.

$$\mathbf{H} = \frac{1}{\sqrt{n}} \begin{bmatrix} 0 & +1 & +1 & +1 & -1 & -1 \\ +1 & 0 & -1 & +1 & +1 & +1 \\ +1 & -1 & 0 & +1 & +1 & +1 \\ +1 & +1 & +1 & 0 & +1 & +1 \\ -1 & +1 & +1 & +1 & 0 & -1 \\ -1 & +1 & +1 & +1 & -1 & 0 \end{bmatrix}$$

It turns out that, as the dimension of the matrix increases, the eigenvalues of the matrix become more and more predictable irrespectively of the exact realization of the matrix. This result enabled to determine the energy levels of many nuclei without considering the very specific nature of the interactions.

In the following, we will provide the different steps of the proof which are of interest for understanding the moment approach.

4.1.2 The moment approach

The moment approach for the derivation of the eigenvalue distribution of random matrices was initiated by the work of Wigner [25]. The main idea is to compute, as the dimension increases, the trace of the matrices $\mathbf{H}^k$ for different exponents k. Although other techniques can be used (via the Beta-function), we review here the historical results. Typically, let

$$dF_N(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i)$$

then the moments of the distribution are given by

$$\begin{aligned} m_1^N &= \frac{1}{N}\text{trace}(\mathbf{H}) = \frac{1}{N}\sum_{i=1}^{N}\lambda_i = \int \lambda dF_N(\lambda) \\ m_2^N &= \frac{1}{N}\text{trace}(\mathbf{H})^2 = \int \lambda^2 dF_N(\lambda) \\ \ldots &= \ldots \\ m_k^N &= \frac{1}{N}\text{trace}(\mathbf{H})^k = \int \lambda^k dF_N(\lambda) \end{aligned}$$

Quite remarkably, as the dimension increases, the traces can be computed using combinatorial approaches involving in particular non-crossing partitions. In this particular case, under some mild assumptions on the distribution of the entries of $\mathbf{H}$, the moments converge to what are known as the Catalan numbers. In particular, we have

$$\begin{aligned} \lim_{N\to\infty}\frac{1}{N}\text{Trace}(\mathbf{H}^{2k}) &= \int_{-2}^{2} x^{2k} f(x)dx \\ &= \frac{1}{k+1}C_k^{2k} \end{aligned}$$

Note that, since the density function of the matrix entries is symmetric, the odd moments of $\mathbf{H}$ vanish. In particular, the only distribution which has all its moments equal to the Catalan numbers is known to be the semi-circular law f given by

$$f(x) = \frac{1}{2\pi}\sqrt{4-x^2}$$

with $|x| \leq 2$. This can be directly verified by recursive calculus:

$$\begin{aligned} \alpha_{2k} &= \frac{1}{\pi}\int_{-2}^{2} x^{2k}\sqrt{4-x^2}dx \\ &= -\frac{1}{2\pi}\int_{-2}^{2}\frac{-x}{\sqrt{4-x^2}}x^{2k-1}(4-x^2)dx \\ &= \frac{1}{2\pi}\int_{-2}^{2}\sqrt{4-x^2}(x^{2k-1}(4-x^2))'dx \\ &= 4(2k-1)\alpha_{2k-2} - (2k+1)\alpha_{2k} \end{aligned}$$

In this way, the recursion is obtained:

$$\alpha_{2k} = \frac{2(2k-1)}{k+1}\alpha_{2k-2}$$

With the additional property that $\alpha_2 = 1$, this recursive sequence can be shown to match the Catalan numbers previously defined.

Let us give another example to understand the moment approach for a single random matrix. Suppose that one is interested in the empirical eigenvalue distribution of $\mathbf{H}\mathbf{H}^H$ where $\mathbf{H}$ is $N \times K$ with i.i.d. Gaussian

entries of variance $\frac{1}{N}$. Additionally we denote $\alpha = \frac{K}{N}$. In this case, in the same manner as above, the moments of this distribution are given by:

$$\begin{aligned} m_1^N &= \frac{1}{N}\text{trace}\left(\mathbf{H}\mathbf{H}^H\right) = \frac{1}{N}\sum_{i=1}^{N}\lambda_i \to 1 \\ m_2^N &= \frac{1}{N}\text{trace}\left(\mathbf{H}\mathbf{H}^H\right)^2 = \frac{1}{N}\sum_{i=1}^{N}\lambda_i^2 \to 1+\alpha \\ m_3^N &= \frac{1}{N}\text{trace}\left(\mathbf{H}\mathbf{H}^H\right)^3 = \frac{1}{N}\sum_{i=1}^{N}\lambda_i^3 \to \alpha^2+3\alpha+1 \end{aligned}$$

It turns out that the only distribution which has the same moments is known to be the Marchenko-Pastur Law.

Example (**Marchenko-Pastur Law**) If $\mathbf{S}$ is an $n\times p$ random matrix which entries are independent centered Gaussian random variables with variance $\frac{1}{p}$, then the eigenvalue distribution of $\mathbf{S}\mathbf{S}^*$ tends to the law μ_λ with density

$$\frac{\sqrt{4\lambda-(x-1-\lambda)^2}}{2\pi\lambda x} \quad \text{on} \quad [(1-\sqrt{\lambda})^2,(1+\sqrt{\lambda})^2]. \tag{12}$$

This law is known as the Marchenko-Pastur law with parameter $1/\lambda$. More precisely, it is the law of λ times a random variable distributed according to the Marchenko-Pastur law with parameter $1/\lambda$.

Remark: In many cases, one would obviously think that the eigenvalues of

$$N\underbrace{\left\{\left[\quad\mathbf{H}\quad\right]\right.}_{K}\left[\quad\mathbf{H}^H\quad\right]$$

when $N\to\infty$, $K/N\to\alpha$ are all equal to one. Indeed, asymptotically, all the diagonal elements are equal to one and the extra-diagonal elements are equal to zero. However, although the matrix tends entry-wise to the identity matrix, the eigenvalue distribution does not tend to a single mass in one. Indeed, the N^2-N extra-diagonal terms tend to zero at a rate $O(\frac{1}{N^2})$. Therefore, the distance of the matrix to the identity matrix in the sense of the Froebenius norm is of order $0(1)$.

The aforementioned moment approach that consists into determining the asymptotic eigenvalue distribution of a large dimensional random matrix based on its moments is often referred to as the *method of moments*. This comes as an alternative to analytical approaches that study distribution functions through analytic properties of their Stieltjes transform, characteristic functions etc. In the subsequent section, we will provide moment approaches for matrix models that cannot be treated to this day based on analytic approaches. However, the matrices under considerations might be such that the moments of their asymptotic eigenvalue distribution does not uniquely determine the underlying distribution; in such a case, the moments will only be useful for what they are and can be used to treat inverse problems (called free deconvolution problems), but will not be useful to infer the asymptotic eigenvalue distribution. For that reason, these techniques will not be referred to as a method of moments, but simply as a moment technique.

4.1.3 Conclusion

The moment technique in statistical physics is very appealing and powerful in order to derive the exact asymptotic moments of random matrices. It requires combinatorial skills and can be used for a large class of random matrices. Our recent work on random Vandermonde matrices [26, 27] has shown again its potential. The main drawback of the technique (compared to other tools such as the Stieltjes transform method) is that it can rarely provide the exact distribution. However, in many applications, one needs only a subset of the moments depending on the number of parameters to be estimated.

4.2 Contributions to the theoretical foundations of statistical physics

4.2.1 Problem addressed: T-nodes deployment

We consider a random deployment of T-nodes and study the quality of the reconstruction field (how the information at the U-node is retrieved) when the signal samples are noisy and randomly positioned. The T-nodes sense the physical phenomenon (hereafter called the field) over the area under observation. The T-nodes can be used to sample a physical field like

- air temperature
- light intensity
- pollution levels
- etc.

and report data to the common processing unit (U-node). The idea is to evaluate the impact of the distribution of the T-nodes on the quality of the reconstructed signal.

For the sake of simplicity (the results have been generalized to the 2-D case in [26, 27]), consider a unidimensional band-limited physical field, described by its harmonics (s is the space index):

$$y(s) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{\frac{-j2\pi ks}{N}}$$

The field is sampled in the space domain by L T-nodes which are deployed in the normalized interval $[0,1]$ in positions $[s_1,...,s_L]$ with $s_i \in [0,1]$ ($\omega_i = \frac{2\pi s_i}{N}$).

$$y(\omega_i) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-jk\omega_i}.$$

The task of the reconstruction algorithm is to calculate an estimate $\hat{\mathbf{x}}$ of the spectrum $\mathbf{x}$ in the presence of noise. In fact, the distribution of the deployment of the sensors affects the performance of signal reconstruction in the presence of noise. In this case, the received signal at the U-node can be written as

$$\mathbf{y} = \mathbf{V}^T\mathbf{x} + \mathbf{n}$$

with $\mathbf{y} = [y(\omega_1),...y(\omega_L)]^T$, $\mathbf{x} = [x_1,...,x_N]^T$,$\mathbf{n} = [n_1,...,n_L]$ and

$$\mathbf{V} = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & \cdots & 1 \\ e^{-j\omega_1} & \cdots & e^{-j\omega_L} \\ \vdots & \ddots & \vdots \\ e^{-j(N-1)\omega_1} & \cdots & e^{-j(N-1)\omega_L} \end{pmatrix}. \quad (13)$$

The usual way of quantifying the error of reconstruction is to define a Minimum Mean Square Error (MMSE) criterion. It turns out that recent results of the interplay between information theory and signal processing [28] have shown that there is a strong connection between the MMSE and the eigenvalues of the matrix $\mathbf{V}\mathbf{V}^T$. In fact, the MMSE is closely related to the empirical eigenvalue distribution of $\mathbf{V}\mathbf{V}^H$ i.e. $dF(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda_i - \lambda)$ where the λ_i are the eigenvalues of $\mathbf{V}\mathbf{V}^H$, by the following relation

$$\text{MMSE} = \frac{1}{N}\mathbb{E}\,||\,\hat{\mathbf{x}} - \mathbf{x}\,||^2 \rightarrow \int g(\lambda)dF(\lambda) \text{(Guo, Verdu, Shamai)}$$

Here, g is a (well behaved) known function.

4.3 Random Vandermonde matrix theory

In [26, 27], we have developed fundamental results for Random Vandermonde matrices $\mathbf{V}$ of dimension $N \times L$ of the form

$$\mathbf{V} = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & \cdots & 1 \\ e^{-j\omega_1} & \cdots & e^{-j\omega_L} \\ \vdots & \ddots & \vdots \\ e^{-j(N-1)\omega_1} & \cdots & e^{-j(N-1)\omega_L} \end{pmatrix} \tag{14}$$

where ω_1,...,ω_L are i.i.d. (phases) taking values on $[0, 2\pi)$.

Without going into details (as it involves sophisticated combinatorial methods), if we define

$$m_n = \lim_{N\rightarrow\infty} E\left[tr_L\left(\left(\mathbf{V}^H\mathbf{V}\right)^n\right)\right]$$

then all the moments m_n of the matrix (and therefore the eigenvalue distribution) can be computed. Quite remarkably, the eigenvalue distribution depends only on the ratio $\frac{L}{N}$ and on the probability distribution of the deployments of the T-nodes. Hence, one can immediately obtain the optimum distribution for a given reconstruction error or the effect of a non-optimal deployment (typically due to some movement) on the reconstruction error. Performance curves and results are available in [26, 27].

5 Paradigms from free probability theory

5.1 Free probability foundations

Free probability [29] theory has grown into an entire field of research dating from the pioneering work of Voiculescu in the 1980's. The basic definitions of free probability are quite abstract, as Voiculescu aimed to introduce an analogy to the independence in classical probability that can be used for non-commutative random variables, such as matrices. These more general random variables are elements in what is called a *non-commutative probability space*. In order not to go into details, we will explain roughly the notion of freeness through a specific example. A detailed introduction can be found in our work [30]. As a starting example, suppose that we know the distribution of the eigenvalues of a matrix $\mathbf{A}$ and of a matrix $\mathbf{B}$. We then inquire on what one can say about the distribution of $\mathbf{C} = \mathbf{A} + \mathbf{B}$. In its full generality, this problem cannot be solved based on trivial operations on the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$. It is actually not enough for the matrices $\mathbf{A}$ and $\mathbf{B}$ to be independent. The exception is when the matrices have the same eigenvectors,

as for diagonal matrices. If this is not the case, it is hard to combine the eigenvectors of **A** and **B** to find the eigenvectors of $\mathbf{A}+\mathbf{B}$. However, in some cases, the asymptotic moments of **C** can be expressed only with the asymptotic moments of **A** and **B**. Indeed, let

$$dF_C^N(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(\lambda-\lambda_1^C\right)$$

If the asymptotic empirical eigenvalue distribution of **C** depends only on the asymptotic empirical eigenvalue distribution of **A** and **B**)

$$m_k^C \quad = \quad \lim_{N\to\infty}\frac{1}{N}\mathrm{Trace}\,(\mathbf{C})^k = f(m_1^A,...,m_k^A,m_1^B,...,m_k^B)$$

then, we say that the matrices are free and the framework falls into the realm of free probability theory. The same holds for the treatment of the product $\mathbf{C}=\mathbf{AB}$. Random variables are elements in what we will call a *non-commutative probability space*: A pair (A,φ), where A is a unital $*$*-algebra* with unit I, and φ is a normalized (i.e. $\varphi(I)=1$) linear functional on A. For matrices, φ is the normalized trace tr_n, defined by

$$tr_n(a) = \frac{1}{n}\sum_{i=1}^{n}a_{ii}.$$

For random matrices, $\varphi=\tau_n$ is defined by

$$\tau_n(a) = \frac{1}{n}\sum_{i=1}^{n}E(a_{ii}) = E(tr_n(a)).$$

The unit in these $*$-algebras is the $n\times n$ identity matrix I_n. The freeness concept is far from the independence concept and there is no link between the two, although independence of the mutual entries of the operand matrices is rather demanded. It assumes some kind of "disconnection" of the eigenvector structure of both matrices. Typically, matrices which are unitarily invariant are free.

5.2 Free probability for information retrieval

A question that naturally arises in BIONETS is the following: "From a set of K noisy and non necessarily equally sampled measurements, what can U-nodes extract in terms of useful information on the network? Moreover, once this information has been extracted, how can the terminal exploit (through dissemination, decision process, etc.) that information?". In this respect, we need to define more precisely the concept of information.

5.2.1 Information and spectrum: how are they related?

The fact that the spectrum of a stationary process is related to the information measure of the underlying process dates back to Kolmogorov [41]. One can show that the entropy rate of a stationary Gaussian stochastic process can be expressed as:

$$H = \log(\pi e) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\log(S(f))df,$$

where S is the spectral density of the process. Hence, if one knows the autocorrelation of the process, one has therefore a full characterization of the information contained in the process. In the discrete case when

considering a random Gaussian vector $\mathbf{x}_i$ of size n, the entropy rate per dimension (or differential entropy) is given by:

$$\begin{aligned} H &= \log(\pi e) + \frac{1}{n}\log\det(\mathbf{R}) \\ &= \log(\pi e) + \frac{1}{n}\sum_{i=1}^{n}\log(\lambda_i), \end{aligned} \tag{15}$$

where $\mathbf{R} = \mathbb{E}(\mathbf{x}_i\mathbf{x}_i^*)$ is the covariance and λ_i the associated eigenvalues. The covariance matrix (and more precisely its eigenvalues) carries therefore all the information of Gaussian networks. The Gaussianity of these networks is due to the fact that the noise, the channel and the signaling are very often Gaussian distributed. Hence, in order to get a reliable estimate of the rate (and in extension the capacity which is the difference between two differential entropies or any other measure which involves performance criteria), one needs to compute the eigenvalues of the covariance matrix. For a number of observations p of the vector $\mathbf{x}_i, i = 1,...,p$, the covariance matrix $\mathbf{R}$ is usually estimated by the empirical covariance matrix of the observed samples,

$$\hat{\mathbf{R}} = \frac{1}{p}\sum_{i=1}^{p}\mathbf{x}_i\mathbf{x}_i^* \tag{16}$$

$$= \mathbf{R}^{\frac{1}{2}}\mathbf{S}\mathbf{S}^*\mathbf{R}^{\frac{1}{2}} \tag{17}$$

Here, $\mathbf{S} = [\mathbf{s}_1,..,\mathbf{s}_p]$ is an $n \times p$ i.i.d. zero mean Gaussian vector with entries of variance $\frac{1}{p}$. It turns out that in wireless random networks, the number of samples p is of the same order as n. This is mainly due to the fact that the network is highly mobile and the statistics are considered to be the same within a p number of samples, which restricts the use of classical processing techniques. Therefore, information retrieval must be performed within a window of limited samples. The main advantage of free deconvolution techniques is that asymptotic behaviours "kick in" at a much earlier stage than other techniques available at the moment. The deconvolution framework comes here from the fact that we would like to invert equation (17) and express $\mathbf{R}$ as a function of $\hat{\mathbf{R}}$. As shown in [31], this is not possible. However, one can compute the eigenvalues of $\mathbf{R}$ knowing only the eigenvalues of $\hat{\mathbf{R}}$ (as the limiting distribution of the eigenvalues of $\mathbf{S}\mathbf{S}^*$ is known to be the Marchenko-Pastur law). This is mainly due to the unitary invariance assumption on one of the matrices (here $\mathbf{S}\mathbf{S}^H$), hence the eigenvector structure does not play a role. The invariance assumption "frees" in some sense one matrix from the other by "disconnecting" their eigenspaces.

5.2.2 Application to the BIONETS framework

The free probability framework has been quite successfully applied in our work [31] to extract information (where the information was solely related to the eigenvalues of the random network) for very simple models, i.e. the case where one of the matrices is unitarily invariant (meaning that we assume some kind of invariance or symmetry in the problem). We were able to compute in each case information measures such as the channel capacity [32] but we are also able to infer on the exact number of T-nodes as well as their positions.

6 Paradigms from Entropy methods

6.1 Maximum Entropy foundations

Maximum Entropy methods have led to a profound theoretical understanding of various scientific areas [36, 37] and has shown the potential of entropy as a measure of our degree of knowledge when encountering a new problem. The principle of maximum entropy[5] is at present the clearest theoretical justification in conducting scientific inference: we do not need a model, entropy maximization creates a model for us out of the available information. Choosing the distribution with greatest entropy avoids the arbitrary introduction or assumption of information that is not available[6]. Bayesian probability theory improves on maximum entropy by expressing some prior knowledge on the model and estimating the parameters of the model. In particular, channel modelling is not a science representing reality but only representing our knowledge upon reality as thoroughly stated by Jaynes in [35]. It answers in particular the following question: based on a given state of knowledge (usually brought by raw data or prior information), what is the best model one can make? This is, of course, a vague question since there is no strict definition of what is meant by "best". But what do we mean then by "best"? Our aim is to derive a model which is adequate with our state of knowledge and which would leave behind arbitrary hypothesis on unknown channel parameters. We need a measure of uncertainty which expresses the constraints of our knowledge and the desire to leave the unknown parameters to lie in an unconstrained space. To this end, many possibilities are offered to us to express our uncertainty. However, we need an information measure which is consistent (complying to certain common sense desiderata, see [38] to express these desiderata and for the derivation of entropy) and easy to manipulate: we need a general principle for translating information into probability assignments. Entropy is the measure of information that fulfills these criteria. Hence, already in 1980, Shore et al. [38] proved that the principle of maximum entropy is the correct method of inference when given new information in terms of expected values. They proved that maximizing entropy is correct in the following sense: maximizing any function but entropy will lead to inconsistencies unless that function and entropy have the same maximum[7]. The consistency argument is at the heart of scientific inference and can be expressed through the following axiom:

Lemma 6.1 *If the prior information* $\mathbf{I_1}$ *on which the channel model* $\mathbf{H_1}$ *is based can be equated to the prior information* $\mathbf{I_2}$ *of the channel model* $\mathbf{H_2}$ *then both models should be assigned the same probability distribution* $P(\mathbf{H}) = P(\mathbf{H_1}) = P(\mathbf{H_2})$.

Any other procedure would be inconsistent in the sense that, by changing indices 1 and 2, we could then generate a new problem in which our state of knowledge is the same but in which we are assigning different probabilities. The consistency property is only one of the required properties for any good calculus of plausibility statement. In fact, R.T Cox in 1946 derived three requirements known as Cox's Theorem[40]:

- Divisibility and comparability: the plausibility of a statement is a real number between 0 (for false) and 1 (for true) and is dependent on information we have related to the statement.

[5] The principle of maximum entropy was first proposed by Jaynes [36, 37] as a general inference procedure when dealing with problems in Physics.

[6] Keynes named it judiciously the principle of indifference [39] to express our indifference in attributing prior values when no information is available.

[7] Thus, aiming for consistency, we can maximize entropy without loss of generality.

- Common sense: Plausibilities should vary with the assessment of plausibilities in the model.
- Consistency: If the plausibility of a statement can be derived in two ways, the two results should be equal.

More precisely, Shore et al. [38] formalize the maximum entropy approach based on four consistency axioms stated as follows[8]:

- Uniqueness: If one solves the same problem twice the same way then the same answer should result both times.
- Invariance: If one solves the same problem in two different coordinate systems then the same answer should result both times.
- System independence: It should not matter whether one accounts for independent information about independent systems separately in terms of different densities or together in terms of a joint density.
- Subset independence: It should not matter whether one treats an independent subset of system states in terms of a separate conditional density or in terms of the full system density.

These axioms are based on the fundamental principle stating that, if a problem can be solved in more than one way, the results should be consistent. Given this statement in mind, the rules of probability theory should lead every person to the same unique solution, provided each person bases his model on the same information. It is noteworthy to say that if a prior distribution Q of the estimated distribution P is available in addition to the expected values constraints, then the principle of minimum cross-entropy (which generalizes maximum entropy) should be applied. The principle states that, of the distribution P that satisfies the constraints, one should choose the one which minimizes the functional:

$$D(P,Q) = \int P(x)\log\left(\frac{P(x)}{Q(x)}\right)dx$$

Minimizing cross-entropy is equivalent to maximizing entropy when the prior Q is a uniform distribution. Intuitively, cross-entropy measures the amount of information necessary to change the prior Q into the posterior P. If measured data are available, Q can be estimated. However, one can only obtain a numerical form for P in this case (which is not always useful for optimization purposes). Moreover, this is not a easy task for multidimensional vectors such as vec($\mathbf{H}$). As a consequence, we will always assume a uniform prior and use therefore the principle of maximum entropy.

6.2 Entropy methods within the BIONETS framework

The problem of modelling the environment is crucial for the efficient design of BIONETS. The wireless channel suffers from many impairments with certain statistics to be discovered. Based on certain *a priori* measurements, the nodes need to model at best the environment in order to schedule/route information adequately, especially in time varying topologies. Using statistical methods we have developed a general framework to model multiple input multiple output environments (this case is very general and not related

[8] In all the rest of the document, the consistency argument will be referred to as Axiom 6.1.

to multiple antennas specifically): It is the whole environment connecting the T-nodes to the U-nodes. We use very limited prior information: only the second order statistics. Interestingly, we able to derive new models for the wireless link by marginalizing the prior distribution using the replica trick (a very sophisticated method of calculation in physics) in [33, 34].

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